A refinement of the Adams' conjecture on theta correspondence

Rui Chen

Based on joint work with Jialiang Zou and joint work with Wee Teck Gan

April 2, 2024

Elementary Scenario

ω: fin-dim rep of finite group $G \times H$. Decompose as G-mod:

$$\omega = \bigoplus_{\pi \in \mathrm{Irr}(G)} \pi \boxtimes \Theta(\pi)$$

$\Theta(\pi)$: multiplicity space of π, inherit H-action from ω. Get:

$$\Theta : K_0(G) \longrightarrow K_0(H),$$

where $K_0(G)$ denotes the Grothendieck group of $\mathrm{Irr}(G)$.

This scenario appears in many places, like: Schur–Weyl duality, Deligne–Lusztig theory, and also theta correspondence.

Weil representation

F: local field of char 0; E: quadratic extension of F. Given:

- Hermitian space V and skew-Hermitian space W;
- auxiliary data (ψ_F, χ_V, χ_W).

Define: $\mathcal{W} = \mathrm{Res}_{E/F}\left(V \otimes_E W\right)$, equipped with:

$$\langle \cdot, \cdot \rangle = \mathrm{tr}_{E/F}\left((\cdot, \cdot)_V \otimes \langle \cdot, \cdot \rangle_W \right).$$

We have a Weil rep ω_{ψ_F} on the $\mathbb{C}^\times$-cover $\widetilde{\mathrm{Sp}}(\mathcal{W})$ of $\mathrm{Sp}(\mathcal{W})$.

By Kudla, (ψ_F, χ_V, χ_W) specifies a splitting:

$$\mathrm{U}(V) \times \mathrm{U}(W) \longrightarrow \widetilde{\mathrm{Sp}}(\mathcal{W}).$$

Let $\omega = \omega_{\psi_F, \chi_V, \chi_W}$ be the pull back of ω_{ψ_F}, called the Weil rep.

Howe duality

For $\pi \in \mathrm{Irr}(\mathrm{U}(W))$, consider maximal π-isotypic quotient of ω:

$$\pi \boxtimes \Theta(\pi),$$

$\Theta(\pi)$: multiplicity space of π, which is a *finite length* rep of $\mathrm{U}(V)$.

Theorem (Howe duality)

1. *If $\Theta(\pi) \neq 0$, then it has a unique irred quotient $\theta(\pi)$.*
2. *Moreover, the map*

$$\theta : \mathrm{Irr}(\mathrm{U}(W)) \backslash \{\pi \mid \theta(\pi) = 0\} \longrightarrow \mathrm{Irr}(\mathrm{U}(V))$$

is injective.

Natural Question: Describe it!

L-parameter and component group

Let $n = \dim_E W$. An L-parameter of $\mathrm{U}(W)$ is an n-dim rep

$$\phi : WD_E \longrightarrow \mathrm{GL}_n(\mathbb{C}),$$

which is *conjugate self-dual* of parity $(-1)^{n-1}$. It is said:

- discrete: if ϕ is multiplicity free;
- tempered: if the W_E has bounded image.

In general, we can write:

$$\phi = \varphi + \sum_i m_i \phi_i + (\varphi^c)^\vee ,$$

where each ϕ_i has parity $(-1)^{n-1}$, and φ is bad parity part. Set:

$$A_\phi = \prod_i \mathbb{Z}/2\mathbb{Z}\, a_i.$$

Local Langlands correspondence

There is a finite to one surjective map:

$$LL : \bigsqcup_{W} \mathrm{Irr}(\mathrm{U}(W)) \longrightarrow \Phi(n),$$

where the disjoint union runs over all n-dim skew-Herm spaces W, and $\Phi(n)$ is the set of L-parameters. For each $\phi \in \Phi(n)$, set:

$$\Pi_{\phi}(\mathrm{U}(W)) = LL^{-1}(\phi) \cap \mathrm{Irr}(\mathrm{U}(W)).$$

Then we have a bijection

$$J : \bigsqcup_{W} \Pi_{\phi}(\mathrm{U}(W)) \longrightarrow \mathrm{Irr}(A_{\phi}).$$

The map LL and J enjoy many good properties, like ECR, LIR...

Some previous works

non-Archimedean case:

- **Atobe–Gan**: describe theta correspondence of *tempered* reps in terms of the LLC.
- **Bakic–Hanzer**: describe theta correspondence of *all* reps based on Atobe–Gan.

Archimedean case:

- **Paul**: (almost) equal rank cases (i.e. $|\dim V - \dim W| \leq 1$).
- **Atobe**: describe non-vanishing of theta correspondence of *tempered* reps.
- **Ichino**: describe theta correspondence of *tempered* reps.

For some other purpose (like global classification, functoriality), need to consider an enlargement of L-packets: A-packets.

Local A-parameter and component group

A local A-parameter of $\mathrm{U}(W)$ is an n-dim rep

$$\psi : WD_E \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C}),$$

which is *conjugate self-dual* of parity $(-1)^{n-1}$. It is said:

- tempered: if $\psi\,|_{\mathrm{SL}_2(\mathbb{C})}$ is trivial and W_E has bounded image.

In general, we can write:

$$\psi = \varphi + \sum_i m_i \psi_i + (\varphi^c)^\vee,$$

where each ψ_i has parity $(-1)^{n-1}$, and φ is bad parity part. Set:

$$A_\psi = \prod_i \mathbb{Z}/2\mathbb{Z}\, a_i.$$

We say ψ is of good parity if $\varphi = 0$.

Working assumptions: local

Since the endoscopic classification of unitary groups has not been completely settled, we shall work under some assumptions:

1. For each ψ, the local A-packet $\Pi_\psi(\mathrm{U}(W))$ is defined; this is a set over $\mathrm{Irr}_{unit}(\mathrm{U}(W))$ equipped with

$$J : \Pi_\psi(\mathrm{U}(W)) \longrightarrow \mathrm{Irr}(A_\psi).$$

2. (LIR) If $\psi = \psi_\tau + \psi_0 + (\psi_\tau^c)^\vee$, then

$$\Pi_\psi(\mathrm{U}(W)) = \left\{ \pi \subset \tau \rtimes \pi_0 \;\middle|\; \pi_0 \in \Pi_{\psi_0}(\mathrm{U}(W_0)) \right\}.$$

   Moreover, if ψ is of good parity, then the NIO $R(\tau \boxtimes \pi_0)$ acts on π by $\epsilon(W)^k J(\pi)(a_\tau)$.

3. $\Pi_\psi(\mathrm{U}(W))$ is multiplicity free.

Global A-parameter and component groups

$\mathbb{F}$: number field, $\mathbb{E}$: quadratic ext. $\mathbb{W}$: skew-Herm space over $\mathbb{E}$.

An A-parameter of $\mathrm{U}(\mathbb{W})$ is a formal sum:

$$\Psi = \rho_1 \boxtimes S_{d_1} + \cdots + \rho_r \boxtimes S_{d_r},$$

where each ρ_i is an cuspidal rep of some $\mathrm{GL}_{k_i}(\mathbb{A}_{\mathbb{E}})$, conjugate self-dual of parity $(-1)^{n+d_i}$, and $k_1 d_1 + \cdots + k_r d_r = n$. If:

$$\rho_i \boxtimes S_{d_i} \neq \rho_j \boxtimes S_{d_j}$$

whenever $i \neq j$, we say Ψ is elliptic. Set:

$$A_\Psi = \prod_i \mathbb{Z}/2\mathbb{Z}\, a_i.$$

Working assumptions: global

1. $L^2_{disc}([\mathrm{U}(\mathbb{W})])$ decomposes into NECs:

$$L^2_{disc}([\mathrm{U}(\mathbb{W})]) = \widehat{\bigoplus_{\Psi}} L^2_{\Psi}([\mathrm{U}(\mathbb{W})]),$$

with each NEC represented by an elliptic A-parameter;

2. for each elliptic A-parameter Ψ, the character $\epsilon_\Psi \in \mathrm{Irr}(A_\Psi)$ is defined, and AMF is established:

$$L^2_{\Psi}([\mathrm{U}(\mathbb{W})]) = \widehat{\bigoplus_{\pi \in \Pi_\Psi(\mathrm{U}(\mathbb{W}), \epsilon_\Psi)}} \pi,$$

where

$$\Pi_\Psi(\mathrm{U}(\mathbb{W}), \epsilon_\Psi) = \left\{ \pi \in {\bigotimes_v}' \Pi_{\Psi_v}(\mathrm{U}(\mathbb{W}_v)) \;\middle|\; J(\pi) = \epsilon_\Psi \right\}.$$

Current status for assumptions

Local (1)(2) and Global (1)(2):

- quasi-split U: by **Mok** (following **Arthur**);
- non quasi-split U:
  - tempered case: by **Kaletha–Minguez–Shin–White**;
  - non-tempered case: onging work by
    - **Kaletha–Minguez–Shin**;
    - **Atobe–Gan–Ichino–Kaletha–Minguez–Shin**.

Local (3):

- non-Archimedean: by **Mœglin**;
- Archimedean: by **Mœglin–Renard**.

The Adams' conjecture

The Adams' conjecture describes theta correspondence in terms of A-parameter. Suppose that

$$n = \dim W \leq m = \dim V.$$

Conjecture (Adams' conjecture)

Let ψ be a local A-parameter of $\mathrm{U}(W)$, and $\pi \in \Pi_\psi(\mathrm{U}(W))$. If $\theta(\pi) \neq 0$, then

$$\theta(\pi) \in \Pi_{\theta(\psi)}(\mathrm{U}(V)),$$

where

$$\theta(\psi) = \psi\chi_V^{-1}\chi_W + \chi_W \boxtimes S_{m-n}.$$

Some previous works

For symplectic-orthogonal dual pair over F non-Archimedean:

Mœglin:

- the Adams' conjecture holds when:

$$m - n \geq \{b - a + 1 \mid \mathbb{1} S_a \boxtimes S_b \subset \psi\}.$$

- the Adams' conjecture not always true.

Bakic–Hanzer:

- for a given A-parameter ψ and $\pi \in \Pi_\psi$, define a number $d(\pi, \psi)$, and show: the Adams' conjecture holds for (π, ψ) whenever $m > d(\pi, \psi)$.
- give an algorithm to compute $d(\pi, \psi)$.

Ingredients of previous works

Explicit construction of A-packets

Mœglin:

1. firstly prove the stable range case (so automatically $\theta(\pi) \neq 0$ and π does not "live on the boudary" of Kudla's filtration).
2. "Descente dans la tour de Witt", generalize the result to a larger range (roughly the largest range so that $\mathbb{1} \boxtimes S_{m-n}$ is the biggest one under some admissible order).

Bakic–Hanzer: Using Mœglin's result as an initial input, produce certain "*candidate*" of the theta lift, and run B. Xu's algorithm.

Remaining questions

There are still many questions one can ask. For example:

1. Given a local A-parameter ψ of $\mathrm{U}(W)$, what can we say about

$$\min \left\{ d(\pi, \psi) \;\middle|\; \pi \in \Pi_\psi(\mathrm{U}(W)) \right\}?$$

2. In the case that Adams' conjecture holds, what is the relation of $J(\theta(\pi))$ and $J(\pi)$?
3. What about unitary dual pairs?

In this talk we will answer (2) and (3) for unitary dual pairs, in the stable range case (i.e. $r_V > n$).

Unlike Mœglin and Bakic–Hanzer, we use a global approach.

Main results

Theorem (C.–Zou)

Suppose $r_V > n$. Let ψ be a local A-parameter of $\mathrm{U}(W)$. Then:

1. *For any $\pi \in \Pi_\psi(\mathrm{U}(W))$, we have $\theta(\pi) \in \Pi_{\theta(\psi)}(\mathrm{U}(V))$.*
2. *The character $\theta(\eta) = J(\theta(\pi))$ can be determined by $\eta = J(\pi)$ as follow:*
   - *if m, n different parity: then $\theta(\eta)\,\big|_{A_\psi} = \eta$;*
   - *if m, n same parity: then $\theta(\eta)(a_i)/\eta(a_i) = \epsilon\left(\frac{1}{2}, \psi_i\chi_V^{-1}, \psi_{E,\delta}\right)$.*
3. *The theta correspondence defines a bijection:*

$$\theta : \bigsqcup_W \Pi_\psi(\mathrm{U}(W)) \longrightarrow \Pi_{\theta(\psi)}(\mathrm{U}(V)),$$

where disjoint union runs over all n-dim skew-Herm spaces.

Key ingredients: local

The notion of low rank unitary reps: introduced by Howe, and extended by J-S. Li.

Locally, there is a bijection:

$$\left(\bigsqcup_W \mathrm{Irr}_{unit}(\mathrm{U}(W))\right) \times \mathrm{Irr}(E^1)$$

$$\Big\downarrow$$

$$\left\{\sigma \in \mathrm{Irr}_{unit}(\mathrm{U}(V)) \;\middle|\; \sigma \textit{ is of rank } n\right\}$$

sending a pair (π, χ) in the above set to $\theta(\pi) \otimes \chi \circ \det$.

Key ingredients: global

Globally, if $\Sigma = \otimes'_v \Sigma_v$ irred unitary rep of $\mathrm{U}(\mathbb{V})$ occuring as a subrep of $\mathcal{A}(\mathrm{U}(\mathbb{V}))$, then the following are equivalent:

1. Σ is of rank n;
2. Σ_v is of rank n for all places v;
3. Σ_v is of rank n for some place v.

Moreover, suppose above conditions hold. Then there exists:

- $\mathbb{W}$: skew-Hermitian space of dim n over $\mathbb{E}$;
- $\Pi = \otimes'_v \Pi_v$: irred unitary rep of $\mathrm{U}(\mathbb{W})(\mathbb{A}_{\mathbb{F}})$;
- χ: automorphic character of $\mathbb{E}^1$,

s.t. $\Sigma = \theta^{abs}(\Pi) \otimes \chi \circ \det$, with $\theta^{abs}(\Pi) = \otimes'_v \theta(\Pi_v)$.

An inequality of J-S. Li

Let Π be an irred unitary rep of $\mathrm{U}(\mathbb{W})(\mathbb{A}_{\mathbb{F}})$. Define:

$$m(\Pi) = \dim \mathrm{Hom}_{\mathrm{U}(\mathbb{W})(\mathbb{A}_{\mathbb{F}})} \left(\Pi, \mathcal{A}(\mathrm{U}(\mathbb{W}))\right),$$

$$m_{disc}(\Pi) = \dim \mathrm{Hom}_{\mathrm{U}(\mathbb{W})(\mathbb{A}_{\mathbb{F}})} \left(\Pi, \mathcal{A}^2(\mathrm{U}(\mathbb{W}))\right).$$

Likewise, can define $m\left(\theta^{abs}(\Pi)\right)$ and $m_{disc}(\theta^{abs}(\Pi))$.

Theorem (J-S. Li)

We have the following inequality:

$$m_{disc}(\Pi) \leq m_{disc}(\theta^{abs}(\Pi)) \leq m(\theta^{abs}(\Pi)) \leq m(\Pi).$$

Sketch of the proof of Theorem (1)

STEP 1: Given ψ: local A-parameter of $\mathrm{U}(W)$ of good parity. Globalize the data:

$$(F, E, \psi, V, W) \rightsquigarrow (\mathbb{F}, \mathbb{E}, \Psi, \mathbb{V}, \mathbb{W}),$$

such that:

- at a place v: $(\mathbb{F}_v, \mathbb{E}_v, \Psi_v, \mathbb{V}_v, \mathbb{W}_v) = (F, E, \psi, V, W)$;
- at a place w: have good control of A_{Ψ_w}.

STEP 2: Given $\pi \in \Pi_\psi(\mathrm{U}(W))$, globalize it to Π using the AMF of $\mathrm{U}(\mathbb{W})$. Applying J-S. Li's inequality, we have:

$$\theta^{abs}(\Pi) \subset L^2_{disc}(\mathrm{U}(\mathbb{V})).$$

Sketch of the proof of Theorem (1)

STEP 3: Determine the A-parameter $\theta(\Psi)$ of $\theta^{abs}(\Pi)$ by doing some unramified computations. Get:

$$\theta(\Psi) = \Psi\chi_{\mathbb{V}}^{-1}\chi_{\mathbb{W}} + \chi_{\mathbb{W}} \boxtimes S_{m-n}.$$

Then localizing at v:

$$\theta(\pi) = \left(\theta^{abs}(\Pi)\right)_v \in \Pi_{\theta(\Psi)_v}\left(\mathrm{U}(\mathbb{V}_v)\right) = \Pi_{\theta(\psi)}(\mathrm{U}(V)).$$

Sketch of the proof of Theorem (2)

non-Archimedean places: we use an idea of **Atobe**:

♠ Instead of π and $\theta(\pi)$, consider all $\chi_V\tau_i \rtimes \pi$ and $\chi_W\tau_i \rtimes \theta(\pi)$.

Here τ_i is the irred unitary rep of some GL.

Gan–Ichino constructed an diagram:

$$\begin{array}{ccc} \widetilde{\omega} \otimes (\chi_W\tau \rtimes \theta(\pi)^\vee) & \longrightarrow & \chi_V\tau \rtimes \pi \\ {\scriptstyle 1\otimes R(\chi_W\tau\boxtimes\theta(\pi)^\vee)}\downarrow & & \downarrow{\scriptstyle R(\chi_V\tau\boxtimes\pi)} \\ \widetilde{\omega} \otimes (\chi_W\tau \rtimes \theta(\pi)^\vee) & \longrightarrow & \chi_V\tau \rtimes \pi \end{array}$$

where $\widetilde{\omega}$ is the Weil rep of some larger groups, and horizontal maps are essentially some Godement–Jacquet integrals. This diagram commutes up to a computable constant. Apply LIR.

Sketch of the proof of Theorem (2)

Archimedean places: similar to the proof of (1), use global method.

STEP 1: Note that in this case $F = \mathbb{R}$ and $E = \mathbb{C}$. Globalize:

$$(F, E, \psi, V, W) \rightsquigarrow (\mathbb{Q}, \mathbb{E}, \Psi, \mathbb{V}, \mathbb{W}),$$

such that at one auxiliary place have good control.

STEP 2: Given $\pi \in \Pi_\psi(\mathrm{U}(W))$, globalize it to Π using the AMF of $\mathrm{U}(\mathbb{W})$. Applying J-S. Li's inequality, we have:

$$\theta^{abs}(\Pi) \subset L^2_{disc}(\mathrm{U}(\mathbb{V})).$$

STEP 3: Using the AMF of $\mathrm{U}(\mathbb{V})$ and (2) of non-Archimedean.

Sketch of the proof of Theorem (3)

To show the surjectivity, for $\sigma \in \Pi_{\theta(\psi)}(\mathrm{U}(V))$, would like to use global method to find its preimage.

- Globalize σ to $\Sigma \subset L^2_{disc}(\mathrm{U}(\mathbb{V}))$, s.t. A-parameter of the form:

$$\theta(\Psi) = \Psi\chi_{\mathbb{V}}^{-1}\chi_{\mathbb{W}} + \chi_{\mathbb{W}} \boxtimes S_{m-n}.$$

- Applying J-S. Li's result, there exists $\Pi \subset \mathcal{A}(\mathrm{U}(\mathbb{W}))$, s.t.

$$\Sigma = \theta^{abs}(\Pi).$$

Issue: Need $\Pi \subset \mathcal{A}^2(\mathrm{U}(\mathbb{W}))$ to apply the AMF!

Sketch of the proof of Theorem (3)

Idea: at one auxiliary place w, suitably choose Σ_w s.t.

- Π_w is strictly negative (i.e. Aubert–Zelevinsky dual of d.s.).

Then by the global square-integrable criterion, $\Pi \subset \mathcal{A}^2(\mathrm{U}(\mathbb{W}))$.
Localizing at v:

$$\pi = \Pi_v \in \Pi_{\Psi_v}(\mathrm{U}(\mathbb{W}_v)) = \Pi_\psi(\mathrm{U}(W)).$$

Turn the tables!

We have shown that: assuming Local (1)(2)(3) and Global (1)(2), then the stable range Adams' conjection holds. In our proof, J-S. Li's inequality is crucial.

Question: can we use the prediction of the Adams' conjecture to deduce some results on the endoscopic classification?

Theorem (C.–Zou)

Let Ψ be an elliptic A-parameter of $\mathrm{U}(\mathbb{W})$, and Π an irred unitary rep of $\mathrm{U}(\mathbb{W})(\mathbb{A}_{\mathbb{F}})$ in the NEC defined by Ψ. Suppose that either:

- *Ψ is tempered; or*
- *$r_{\mathbb{W}} \leq 1$.*

Then we have $m_{disc}(\Pi) = m(\Pi)$.

Turn the tables!

Now only assume Local (1)(2)(3) and Global (1)(2) for q-split U.

$\mathbb{F}$: number field, $\mathbb{E}$: quadratic ext. $\mathbb{W}$: skew-Herm space over $\mathbb{E}$.

- Take $\mathbb{V}$ split Herm space over $\mathbb{E}$, s.t. $r_{\mathbb{V}} > \dim \mathbb{W}$, and $\dim \mathbb{V}$ has different parity with $\dim \mathbb{W}$.
- Locally, define:

$$\Pi_{\Psi_v}(\mathrm{U}(\mathbb{W}_v)) = \left\{\pi \in \mathrm{Irr}_{unit}(\mathrm{U}(\mathbb{W}_v)) \;\middle|\; \theta(\pi) \in \Pi_{\theta(\Psi_v)}(\mathrm{U}(\mathbb{V}_v))\right\}.$$

- Globally, define $\epsilon_\Psi = \epsilon_{\theta(\Psi)}\big|_{A_\Psi}$, and:

$$\Pi_\Psi(\mathrm{U}(\mathbb{W}), \epsilon_\Psi) = \left\{\pi \in {\bigotimes_v}' \Pi_{\Psi_v}(\mathrm{U}(\mathbb{W}_v)) \;\middle|\; J(\pi) = \epsilon_\Psi\right\}.$$

Under the condition of above theorem, J-S. Li's inequality is an equality. We obtain the AMF of $\mathrm{U}(\mathbb{W})$ from that of $\mathrm{U}(\mathbb{V})$.

Turn the tables!

Theorem

1. *$L^2_{disc}([\mathrm{U}(\mathbb{W})])$ decomposes into NECs:*

$$L^2_{disc}([\mathrm{U}(\mathbb{W})]) = \widehat{\bigoplus_{\Psi}} L^2_{\Psi}([\mathrm{U}(\mathbb{W})]),$$

with each NEC represented by an elliptic A-parameter;

2. *Suppose that either Ψ is tempered, or $r_{\mathbb{W}} \leq 1$. Then*

$$L^2_{\Psi}([\mathrm{U}(\mathbb{W})]) = \widehat{\bigoplus_{\pi \in \Pi_{\Psi}(\mathrm{U}(\mathbb{W}), \epsilon_{\Psi})}} \pi.$$

Remark: This idea was first used by Gan–Ichino to study Mp_{2n}.

Application: special case of twisted GGP

F: local field of char 0; E, K: quadratic extension of F, s.t.

$$L = E \otimes_F K$$

is a biquadratic extension. W: n-dim skew-Herm space over E. The twisted GGP problem concerns about:

$$\mathrm{Hom}_{\mathrm{U}(W)}(\pi, \omega)$$

for $\pi \in \mathrm{Irr}(\mathrm{U}(W_K))$. Here ω: Weil rep of $\mathrm{U}(\ell_1) \times \mathrm{U}(W)$.

By the Adams' conjecture, the A-parameter of ω is of the form:

$$\chi + \mu \boxtimes S_{n-1}.$$

Application: special case of twisted GGP

If $\mathcal{V}$: $(n-1)$-dim Herm space over L; $\sigma \in \mathrm{Irr}(\mathrm{U}(\mathcal{V}))$, s.t.

$$\pi = \theta(\sigma).$$

Consider the seesaw diagram:

$$\begin{tikzcd}
\mathrm{U}(W_K) \arrow[d, dash] \arrow[drr, dash] & & \mathrm{U}(\mathrm{Res}_{L/E}\mathcal{V}) \arrow[d, dash] \arrow[dll, dash] \\
\mathrm{U}(W) & & \mathrm{U}(\mathcal{V})
\end{tikzcd}, \qquad
\begin{tikzcd}
\pi = \theta(\sigma) \arrow[d, dash] \arrow[drr, dash] & & \theta(\sigma) \arrow[d, dash] \arrow[dll, dash] \\
\omega & & \sigma
\end{tikzcd},$$

By the Adams' conjecture the A-parameter of $\theta(\omega)$ is of the form:

$$\chi + \mu \boxtimes S_{n-1} + \lambda \boxtimes S_{n-2}.$$

So it comes from some ω_0: Weil rep of $\mathrm{U}_1 \times \mathrm{U}_{n-1}$!

Application: special case of twisted GGP

W_0: $(n-1)$-dim skew-Herm space over E determined by theta dichotomy. Using a similar seesaw diagram

$$\begin{tikzcd}
\mathrm{U}(\mathrm{Res}_{L/E}\mathcal{V}) \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}(W_{0,K}) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{U}(\mathcal{V}) & \mathrm{U}(W_0)
\end{tikzcd},$$

this allow us to reduce the case to twisted GGP of $\mathrm{U}(W_0)$.

Theorem (C.–Gan)

Let ϕ be an L-parameter of $\mathrm{U}(W)$ of the form

$$\chi_1 + \chi_2 + \cdots + \chi_n.$$

Then the twisted GGP conjecture holds for ϕ.

Further questions

1. For irred unitary reps lying in the NEC of an elliptic A-parameter, is J-S. Li's inequality always an equality?
2. Gan–Ichino has studied the tempered automorphic spectum of Mp_{2n}, what about the non-tempered spectrum?
3. Following Mœglin, one can define local A-packets of Mp_{2n} using stable range theta correspondence $\mathrm{Mp}_{2n} \times \mathrm{SO}_{2r+1}$. Recently there are many works on explicit construction of A-packets of SO_{2r+1}. Can we transfer those results to Mp_{2n}?

Thank you for your attention!